

Notes on Representing \aleph_0 -categorical Linear Orders

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Abstract

These notes find a canonical representation of the \aleph_0 -categorical linear orders based on Joseph Rosenstein's description. A unique minimal representation, called the normal form, is obtained.

The \aleph_0 -categorical linear orders were classified by Joseph Rosenstein in [2], where he constructs them from finite linear orders using two operations.

Definition 1 If $\langle L_0, <_0 \rangle$ and $\langle L_1, <_1 \rangle$ are linear orders then their concatenation, denoted by $L_0 \wedge L_1$ is the linear order $\langle L_0 \cup L_1, < \rangle$, where

$$x < y \quad \text{iff} \quad \begin{cases} (x, y \in L_0 \quad \text{and} \quad x <_0 y) & \text{or} \\ (x, y \in L_1 \quad \text{and} \quad x <_1 y) & \text{or} \\ (x \in L_0 \quad \text{and} \quad y \in L_1) \end{cases}$$

Definition 2 $\langle \mathbb{Q}_n, <_{\mathbb{Q}_n}, C_1 \dots C_n \rangle$ is the Fraïssé generic n -coloured partial order, i.e. the countable dense linear order with n colours which occur interdensely (for all x and y there are z_1, \dots, z_n between x and y such that $C_i(z_i)$ holds for each i).

\mathbb{Q}_n is the Fraïssé limit of n -coloured linear orders, and hence is \aleph_0 -categorical.

Definition 3 Let $\langle L_1, <_1 \rangle, \dots, \langle L_n, <_n \rangle$ be linear orders. For each $q \in \mathbb{Q}_n$ we define $L(q)$ to be a copy of $\langle L_i, <_i \rangle$ if $\mathbb{Q}_n \models C_i(q)$. The \mathbb{Q}_n -shuffle of $\langle L_1, <_1 \rangle, \dots, \langle L_n, <_n \rangle$, denoted by $\mathbb{Q}_n(L_1, \dots, L_n)$, is the linear order $\langle \bigcup_{q \in \mathbb{Q}_n} L(q), < \rangle$, where

$$x < y \quad \text{iff} \quad \begin{cases} x, y \in L(q) & \text{and} \quad x <_i y & \text{or} \\ x \in L(q), y \in L(p) & \text{and} \quad q <_{\mathbb{Q}_n} p \end{cases}$$

Theorem 4 (Rosenstein) L is an \aleph_0 -categorical linear order if and only if L can be constructed from singletons by a finite number of concatenations or shuffles.[1]

This leads to a formal representation of the \aleph_0 -categorical linear orders

Definition 5 A term is built as follows:

Singleton The singleton 1 is a term

Concatenation If t_0, t_1 are terms then $t_0 \wedge t_1$ is a term.

\mathbb{Q}_n -shuffle If t_0, \dots, t_{n-1} are terms then $\mathbb{Q}_n(t_0, \dots, t_{n-1})$ is a term.

How terms represent linear orders is obvious. If t is a term, then let L_t be the linear order represented by t . We say that a term t is a shuffle if there are t_0, \dots, t_{n-1} such that $t = \mathbb{Q}_n(t_0, \dots, t_{n-1})$.

Concatenation is obviously associative, so we will not bother using brackets. The terms can be interpreted as linear orders in the obvious way. Note that multiple terms can denote the same linear order.

Example 6 The rationals can be represented by both $\mathbb{Q}_1(1)$ and $\mathbb{Q}_2(1, 1)$.

This leads to the question of which terms represent the same \aleph_0 -categorical linear order, and if there is a canonical choice of representative. We also discuss a slightly wider class of linear orders by allowing infinite concatenation, which will be dealt with by discussing infinite sequences of terms. While their concatenation will not necessarily be an \aleph_0 -categorical linear order we will be able to arrive at a canonical representation of these as well.

Lemma 9 lists the ways in which terms can represent the same linear order. These all concern terms which encode infinite linear orders, as the representation for finite linear orders is automatically unique.

Definition 7 The **complexity** of a term T , written as $c(T)$, is the number of concatenations and shuffles in the term added to the sum of all the lengths of the shuffles contained in the term. The complexity of a sequence of terms is the sum of the complexities of all of the terms that appear in the sequence and the length of the sequence.

Note that this means that all infinite sequences have infinite complexity.

Definition 8 The **depth** of a term T , written as $d(T)$, is defined as follows:

- If T represents a finite linear order then $d(T) = 0$.
- If T is of the form $\mathbb{Q}_n(t_0, \dots, t_{n-1})$ then $d(T) = \max(d(t_i)) + 1$.
- If T is of the form $s_0 \wedge \dots \wedge s_{n-1}$ then $d(T) = \max(d(s_j))$.

Lemma 9 Let t_0, \dots, t_{n-1} be terms, let $m \leq n$ and let f be a permutation of n . We also let τ be either the empty set or one of the t_i . Then the following are isomorphic to $\mathbb{Q}_n(t_0, t_1, \dots, t_{n-1})$:

1. $\mathbb{Q}_n(t_{f(0)}, t_{f(1)}, \dots, t_{f(n-1)})$;
2. $\mathbb{Q}_{n+1}(t_0, \dots, t_{n-1}, t_m)$;
3. $\mathbb{Q}_{m+1}(t_0, \dots, t_{m-1}, \tau_0 \wedge \mathbb{Q}_n(t_0, \dots, t_{n-1}) \wedge \tau_1)$ where $\tau_0, \tau_1 \in \{\emptyset, t_0, \dots, t_{n-1}\}$; and
4. $\mathbb{Q}_m(t_0, \dots, t_{m-1}) \wedge \tau \wedge \mathbb{Q}_m(t_0, \dots, t_{m-1})$ where $\tau \in \{\emptyset, t_0, \dots, t_{m-1}\}$.

Proof

The roles of the colours in \mathbb{Q}_n are interchangeable, so

$$\mathbb{Q}_n(t_0, \dots, t_{n-1}) \cong \mathbb{Q}_n(t_{f(0)}, t_{f(1)}, \dots, t_{f(n-1)})$$

The structure $\langle \mathbb{Q}_{n+1}, <_{\mathbb{Q}_n}, C_1, \dots, C_{m-1}, C_m \vee C_{n+1}, C_{m+1}, \dots, C_n \rangle$ is a countable dense linear order with n colours which occur interdensely, and therefore

$$\langle \mathbb{Q}_{n+1}, <, C_0, \dots, C_{m-1}, C_m \vee C_n, C_{m+1}, \dots, C_{n-1} \rangle \cong \langle \mathbb{Q}_n, <, C_0, \dots, C_{n-1} \rangle$$

and $\mathbb{Q}_{n+1}(t_0, \dots, t_{n-1}, t_m) \cong \mathbb{Q}_n(t_0, t_1, \dots, t_{n-1})$.

The structure $\mathbb{Q}_{m+1}(t_0, \dots, t_{m-1}, \tau_0 \wedge \mathbb{Q}_n(t_0, \dots, t_{n-1}) \wedge \tau_1)$ is obtained by replacing the C_i coloured elements of \mathbb{Q}_{m+1} by t_i if $i < m - 1$ or by $\tau_0 \wedge \mathbb{Q}_n(t_0, \dots, t_{n-1}) \wedge \tau_1$ if $i = m$. The structure $\mathbb{Q}_n(t_0, \dots, t_{n-1})$ is obtained by replacing the C_i coloured elements of \mathbb{Q}_n by t_i . Let M be the coloured linear order obtained by replacing the C_m coloured elements of \mathbb{Q}_{m+1} by $x \wedge \mathbb{Q}_n \wedge y$, where x and y are coloured according to the values taken by $\tau, \sigma \in \{\emptyset, t_0, \dots, t_{n-1}\}$. (If $\tau = \emptyset$ then we delete x .) We may also obtain $\mathbb{Q}_n(t_0, \dots, t_{n-1})$ by replacing the C_i coloured elements of M by t_i . This M is a dense linear order in which the colours C_0, \dots, C_{n-1} occur interdensely, and so $M \cong \mathbb{Q}_n$ and hence

$$\mathbb{Q}_n(t_0, \dots, t_{n-1}) \cong \mathbb{Q}_{m+1}(t_0, \dots, t_{m-1}, \tau_0 \wedge \mathbb{Q}_n(t_0, \dots, t_{n-1}) \wedge \tau_1)$$

Let M be the structure $\mathbb{Q}_n \wedge \{x\} \wedge \mathbb{Q}_n$ where x is coloured by C_i if and only if $\tau = t_i$. The structure $\mathbb{Q}_n(t_0, \dots, t_{n-1}) \wedge \tau \wedge \mathbb{Q}_n(t_0, \dots, t_{n-1})$ can be obtained by replacing the C_i coloured elements of M by t_i , however M is a dense coloured linear order where the colours C_i for $i < n$ occur interdensely, and so $M \cong \mathbb{Q}_n$. Therefore

$$\mathbb{Q}_n(t_0, \dots, t_{n-1}) \cong \mathbb{Q}_n(t_0, \dots, t_{n-1}) \wedge \tau \wedge \mathbb{Q}_n(t_0, \dots, t_{n-1})$$

□

Definition 10 We use induction over the formation of terms to define when a term is in **normal form** (n.f.).

1. All finite terms are in n.f.
2. A term of the form $\mathbb{Q}_m(t_0, \dots, t_{m-1})$ is in n.f. if:
 - (a) all the t_i are in n.f.; and
 - (b) Numbers 2 and 3 of Lemma 9 do not apply.

If the t_i are permuted then the term is unaltered.

3. A term of the form $t_0 \wedge \dots \wedge t_{n-1}$ is in n.f. if all the t_i are in n.f. and no $t_{i-1} \wedge t_i \wedge t_{i+1}$ or $t_i \wedge \emptyset \wedge t_{i+1}$ satisfy Number 4 of Lemma 9.

A possibly infinite sequence of terms (s_i) is said to be in **normal form** if:

1. each s_i is in normal form;
2. no $s_{i-1} \wedge s_i \wedge s_{i+1}$ or $s_i \wedge \emptyset \wedge s_{i+1}$ satisfy Number 4 of Lemma 9;
3. if s_j is finite either:
 - (a) s_{j+1} is infinite; or
 - (b) (s_i) is an infinite sequence and $s_j = s_k = 1$ for all $k \geq j$.

If (s_i) is a sequence in normal form, and L is a linear order represented by (s_i) then we say that (s_i) is the n.f. representation of L .

Definition 11 Let t be a shuffle, and let s be a term shuffled by t . Then we say that $\sigma \in \alpha(s, t)$ if $\sigma \subseteq L_t$ is obtained in the construction of L_t by replacing an element of \mathbb{Q}_n by L_s .

Lemma 12 Let S and T be shuffles such that $L_S \cong L_T$, witnessed by ϕ . Suppose S shuffles an s such that for all t shuffled by T and all $\sigma \in \alpha(s, S)$

$$\phi(\sigma) \notin \alpha(t, T)$$

Then L_s is isomorphic to $A_0 \wedge L_S \wedge A_1$, where A_0 (resp. A_1) is either equal to a terminal (resp. initial) segment one of the terms that T shuffles or empty.

Proof

L_T is obtained by replacing the points of \mathbb{Q}_n with the appropriate linear order. Let χ be the map from T to \mathbb{Q}_n that sends a point in L_T to the point in \mathbb{Q}_n that t was obtained from when building L_T .

Since σ is a bounded convex subset of L_S and these properties are preserved by the maps ϕ and χ we know that $\chi(\phi(\sigma))$ is a bounded interval. We denote the interior of $\chi(\phi(\tilde{s}))$ by I .

$\chi^{-1}(I)$ is contained in $\phi(\sigma)$ and since open intervals of \mathbb{Q}_n are isomorphic to \mathbb{Q}_n we have that $\chi^{-1}(I)$ is isomorphic to L_T . We now define two sets

- $A_0 := \{x \in \phi(\tilde{s}) : x < \chi^{-1}(I)\}$

- $A_1 := \{x \in \phi(\tilde{s}) : x > \chi^{-1}(I)\}$

A_0 must be contained in a copy of some term that T shuffles (say t_0), as otherwise there would be an $x \in A_0$ such that $\chi(x) \in I$. Since $\phi(\sigma)$ is convex, A_0 must be a terminal segment of some $\tau_0 \in \alpha(t_0, T)$. Similarly A_1 must be an initial segment some $\tau_1 \in \alpha(t_1, T)$. It is here that we note that $\phi(\sigma) \cong A_0 \wedge \chi^{-1}(I) \wedge A_1$, as required \square

Proposition 13 *Let S and T be terms in normal form. If $L_S \cong L_T$ then $S = T$.*

Proof

Let $\phi : L_S \rightarrow L_T$ be an isomorphism. We assume that $d(T) \leq d(S)$. If L_S and L_T are finite then they are automatically represented by the same term, so we now assume that L_S and L_T are infinite. Let $T = \mathbb{Q}_n(t_0, \dots, t_{n-1})$ and $S = \mathbb{Q}_m(s_0, \dots, s_{m-1})$.

We will prove this by induction on the depth of the shuffle of T , with both stages of the induction also proved by inducting on the depth of the shuffle of S . Let ϕ be an isomorphism from L_S to L_T . Note that in the base case $d(T) = 1$ means that every member of each $\alpha(t_i, T)$ is finite, and so can be recognised as the maximal finite convex subsets of L_T .

$d(T) = 1$ and $d(S) = 1$. The base case for the induction on the depth of the shuffle of T will be proved by another induction, this time on the depth of S . Let $d(S) = 1$, i.e. each s_j is finite. For every $\tau \in \alpha(t_i, T)$ the image $\phi(\tau) \in \alpha(s_j, S)$ for some s_j , and for every $\sigma \in \alpha(s_j, S)$ the preimage $\phi^{-1}(\sigma) \in \alpha(t_i, T)$ for some t_i . This means that for every t_i there is an s_j such that $t_i = s_j$ and vice versa. Since T and S are in normal form both m and n are minimal. This means that $n = m$ and (t_0, \dots, t_{n-1}) is a permutation of (s_0, \dots, s_{m-1}) and $T = S$.

$d(T) = 1$ and $d(S) = p$ where $p > 1$. By the induction hypothesis if U is a n.f. term such that $d(U) < p$ and if L_U is isomorphic to L_T then $T = U$. Since $d(S) > 1$ at least one of the s_i must be infinite. Let s_j be one of the infinite s_i . Since s_j is infinite, $\phi^{-1}(\sigma) \notin \alpha(t_i, T)$ for all $\sigma \in \alpha(s_j, S)$. Lemma 12 shows that $\tilde{s}_j \cong A_0 \wedge L_T \wedge A_1$.

If A_1 is a proper terminal segment of τ then there are $x, y \in \tau$ such that $\phi^{-1}(x)$ is not contained in some $\sigma \in \alpha(s_j, S)$, but $\phi^{-1}(y)$ is. This implies that the interval $[x, y]$ is finite while the interval $[\phi^{-1}(x), \phi^{-1}(y)]$ is infinite, contradicting the fact that ϕ is an isomorphism, so $\tau \in \{\emptyset\} \cup \alpha(t_0, T)$. Similarly A_2 is either empty or contained in $\alpha(t_1, T)$. From this we can conclude that $s_j = t_0 \wedge s'_j \wedge t_1$ where $L_{s'_j} \cong L_T$.

Since $d(s'_j) < p$ we know that $s'_j = T$. This means that every s_j is either equal to a t_i or is of the form $\sigma \wedge T \wedge \tau$ contradicting that both S and T are in normal form.

$d(T) = p$ and $d(S) = p$ where $p > 1$. Now suppose that if U and U' are in n.f. and $d(U) < p$ then $L_U \cong L_{U'} \Rightarrow U = U'$. Pick an s_j , and let $\sigma \in \alpha(s_j, S)$. We know that $\phi(\sigma)$ must be entirely contained in one of the \tilde{t}_i , as otherwise we would be able to apply Lemma 12 and find that σ is isomorphic to $A_0 \wedge L_T \wedge A_1$, which implies that L_T is isomorphic to something of depth less than p , giving a contradiction.

So $\phi(\sigma)$ is entirely contained some $\tau \in \alpha(t_i, T)$. Similarly $\phi^{-1}(\tau)$ must be entirely contained in an element of $\alpha(s_j, S)$. Hence $\phi(\sigma) = \tau$ and $t_i = s_j$. Therefore every term shuffled by S is equal to a term shuffled by T and vice versa. Since both S and T are in normal form there can be no repeated terms in the shuffle, so $L_T = L_S$.

$d(T) = p$ and $d(S) = r$ where $r > s$. Now suppose that L_S has depth r where $p < r$, and for all U and U' in n.f. if $d(U) < r$ then $L_U \cong L_{U'} \Rightarrow U = U'$. There is a $\sigma \in \alpha(s_j, S)$ such that $\phi(\sigma)$ is not entirely contained in any element of any $\alpha(t_i, T)$ in which case $\sigma \cong A_0 \wedge L_T \wedge A_1$. The depth of every t_i is less than r and so $t_i = \sigma \wedge T \wedge \tau$.

Suppose A_0 is a proper non-empty subset of an element of $\alpha(t_0, T)$. This implies that $\phi^{-1}(t_0)$ is not contained entirely in one of the terms shuffled by S and so by the Lemma 12, $t_0 \cong B_0 \wedge L_S \wedge B_1$. As in

the previous step this results in a term with depth less than p being isomorphic to L_S , contradicting the induction hypothesis. This means that A_0 is either empty or equal to t_0 . Similarly A_1 is either empty or equal to t_1 . Since s_j has depth less than r it must be equal to $t_0 \wedge T \wedge t_1$. Therefore every term shuffled by S is either a term shuffled by T , or equal to $t_0 \wedge T \wedge t_1$, and hence S is not in normal form. \square

Proposition 14 *If (T_i) and (S_i) are sequences in normal form whose linear orders encoded by their concatenations are isomorphic then $T_j = S_j$ for all j .*

Proof

Let L_i and M_i be the linear orders encoded by T_i and S_i respectively and let L and M correspond to the concatenations. Let $\phi : L \rightarrow M$ be an isomorphism. If T_0 is finite then ϕ must map L_0 to M_0 as automorphisms preserve least elements. If L_0 is infinite suppose that i is the greatest number such that an element of L_0 is mapped to M_i .

If $i = 1$ then M_1 cannot be finite as L_0 does not have a maximal element. Since neither M_0 nor M_1 have maximal or minimal elements $\phi^{-1}(M_0)$ and $\phi^{-1}(M_1)$ do not have maximal or minimal elements, hence they are unbounded open intervals of L_0 and isomorphic to L_0 . However (M_i) is in normal form, so M_0 and M_1 cannot be isomorphic.

If $i > 1$ then we consider $\phi^{-1}(M_0) \wedge \dots \wedge \phi^{-1}(M_i)$. The argument of the previous paragraph can be adapted to show that M_0 and $M_i \cong L_0$. Since L_0 is a shuffle, M_0 does not have a maximal element and M_i does not have a minimal element. We may assume that for $0 < j < i$ none of the M_j 's are isomorphic to L_0 , as otherwise we can consider M_0, \dots, M_j instead of M_0, \dots, M_i .

Since none of the M_j are isomorphic to L_0 we know that $\phi^{-1}(M_1 \wedge \dots \wedge M_{i-1})$ is contained in a copy of one of the terms that L_0 shuffles. Moreover the ϕ -preimage of $(M_1 \wedge \dots \wedge M_{i-1})$ must be a copy of one of the terms that L_0 shuffles as ϕ is an isomorphism and being strictly contained in a copy of a term would prevent ϕ from being a surjection. This means that we are able to apply Lemma 9 to $M_0 \wedge \dots \wedge M_i$, contradicting the assumption that (M_i) is in normal form.

From this we deduce that $L_0 \cong M_0$, and repeating this argument shows that $L_j \cong M_j$ for all j . Therefore any automorphism between L and M must send L_j to M_j . Since (L_i) and (M_j) are both in normal form, the singletons L_j and M_j are also in normal form and are represented by the same term by the preceding lemma, proving that the sequences (T_i) and (S_i) are equal. \square

So now we have that if a term or a sequence of terms has a normal form representation then this representation is unique and that different normal form sequences represent different linear orders. Finally we need to show that every sequence of terms has a normal form sequence that encodes the same linear order. We shall first show that all finite sequences of arbitrary length have a normal form representation before considering infinite sequences.

Proposition 15 *If t is a term that is not in normal form then there is a term s such that $L_t = L_s$ and s is in normal form. Furthermore, $c(s) < c(t)$.*

Proof

We prove this by induction on the complexity of terms. All finite terms are in normal form, so the base case is immediate. Suppose that if $c(s) < n$ then there is an s' such that $L'_s = L_s$ and s' is in normal form, and $c(s') < c(s)$. Let t be such that $c(t) = n$ and t is not in normal form.

t is not finite, as t is not in normal form.

Suppose that $t = Q_n(t_0, \dots, t_{n-1})$. If t_i is not in normal form, then let t'_i be such that $L_{t_i} = L_{t'_i}$. Let $t' := Q_n(t_0, \dots, t'_i, \dots, t_{n-1})$. Since $c(t'_i) < c(t_i)$, we know that $c(t') < c(t)$, and so $c(t')$ is either in normal form, in which case we are done, or by the induction hypothesis there is a t'' which is in normal form, $c(t'') < c(t')$ and $L_{t''} = L_{t'}$.

Suppose that $t = t_0 \wedge \dots \wedge t_{n-1}$. If there is an i such that $t_i \wedge t_{i+1} \wedge t_{i+2}$ represents the same linear order as t_i then we can replace $t_i \wedge t_{i+1} \wedge t_{i+2}$ by t_i to obtain a term t' such that $L_t = L_{t'}$ and $c(t') < c(t)$. Similarly,

if there is an i such that $t_i \wedge \emptyset \wedge t_{i+1}$ represents the same linear order as t_i then we can replace $t_i \wedge \emptyset \wedge t_{i+1}$ by t_i to obtain a term t' such that $L_t = L_{t'}$ and $c(t') < c(t)$. \square

Proposition 16 *If (t_i) is an infinite sequence of terms there is a sequence (s_i) such that (s_i) represents the same linear order as (t_i) and is in normal form.*

Proof

If we are given a sequence of terms (t_i) we can construct a normal form sequence (s_j) which represents the same linear order inductively. We first let s_0^0 be the normal form representative of t_0 . Suppose that we have considered t_i for $i < n$, obtaining the sequence (s_j^{n-1}) for $j < m$. Let t'_n be the normal form term representing the same linear order as t_n . We construct (s_j) as follows:

1. If both s_{m-1}^{n-1} and t'_n are finite, but there is some t_k which is a \mathbb{Q}_n -shuffle for $k > n$, then let $s_{m-1}^n = s_{m-1}^{n-1} \wedge t'_n$ and $s_i^n = s_i^{n-1}$ for $i < n$. We then consider t_{n+1} .
2. If both s_{m-1}^{n-1} and t'_n are finite, and there is no t_k which is a \mathbb{Q}_n -shuffle for $k > n$, then if $j < m-1$ let $s_j^n = s_j^{n-1}$. Otherwise, let $s_i^\omega = 1$ for all $i \geq m$ and stop.
3. If $s_{m-2} \wedge s_{m-1} \wedge t'_n$ satisfies Number 4 of Lemma 9, then let $s_{m-1}^n = \emptyset$ and $s_i^n = s_i^{n-1}$ for $i < m$. We then consider t_{n+1} .
4. If $s_{m-1} \wedge \emptyset \wedge t'_n$ satisfies Number 4 of Lemma 9, then let $s_i^n = s_i^{n-1}$ for $i < m$. We then consider t_{n+1} .
5. Otherwise let $s_m^n = t'_n$ and $s_i^n = s_i^{n-1}$ for $i < m$. We then consider t_{n+1} .

It is easy to see that each sequence (s_i^n) represents the same linear order as $(t_i)_{i=0}^n$. If this process never terminates, let s_i^ω be the term taken by the tail of the sequence $(s_i^n)_{n \in \mathbb{N}}$. If that sequence has no constant tail then let $s_i^\omega = \emptyset$. By construction, the sequence (s_i^ω) is in normal form.

If we obtain (s_i^ω) via Number 2 then (s_i^ω) represents the same linear order as (t_i) . Suppose that we obtain (s_i^ω) as a limit. Suppose that $s_{j+1}^\omega \neq \emptyset$. Let m_j be the least number such that $s_{j+1}^n \neq \emptyset$ for all $m > n$. Then $(s_i^\omega)_{i=0}^j$ represents the same linear order as $(t_i)_{i=0}^{m_j-1}$. Therefore s_j^ω represents the same linear order as $(t_i)_{i=0}^{m_j-1}$, and if (s_i^ω) is an infinite sequence then it represents the same linear order as (t_i) .

Suppose that (s_i^ω) terminates, and let s_j^ω be the final element. Then eventually we always eventually apply Number 3 to $s_j^n \wedge s_{j+1}^n \wedge t'_n$ or Number 4 to $s_j^n \wedge t'_n$ for some n . Therefore s_j^ω is isomorphic to the linear order represented by the appropriate tail of (t_i) , and (s_i^ω) represents the same linear order as (t_i) . \square

References

- [1] Joseph G. Rosenstein *Linear Orderings* Academic Press 1982 pp. 139-141, 298-299.
- [2] Joseph G. Rosenstein \aleph_0 -categoricity of *Linear Orderings*, Fund. Math 64 (1969), 1-5